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Journal of Computational and Applied Mathematics 178 (2005) 305–312
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

An extension of the Chebyshev polynomials

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Received 2 October 2003; received in revised form 21 June 2004

Abstract

We consider a generalization of the Chebyshev polynomials of the second kind. These polynomials can be expressed as transformed Chebyshev polynomials of a complex variable. The idea of this generalization comes from an extension of typically real functions, where as the kernel function appears the q-Koebe functions proposed by Gasper (SIAM J. Math. Anal. 20 (1989) 1019), which play the role of generating function.

Keywords: Chebyshev polynomials of the first and the second kind; Typically real functions

1. Introduction

In one of his papers, Gasper [4] proved several interesting q-extensions of the inequalities used in [1] in de Branges' proof of the Milin–Bieberbach conjectures. Gasper [4] proposed to study some q-extensions of the Löwner differential equation and suggested that the important role of the Koebe function

$$k_1(z) = \frac{z}{(1-z)^2} = z \cdot {}_1F_0\left[\begin{matrix}2\\ -\end{matrix}; z\right], \quad z \in \mathbb{D} = \{z : |z| < 1\} \tag{1}$$

should be played by the q-Koebe function

$$k_q(z) = \frac{z}{(1-z)(1-qz)} = z \cdot {}_1\phi_0\left[\begin{matrix}q^2\\ -\end{matrix}; q, z\right], \quad z \in \mathbb{D}. \tag{2}$$

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0377-0427/$ - see front matter © 2004 Published by Elsevier B.V.
doi:10.1016/j.cam.2004.06.021

The parameter q here and below belongs to the interval $(-1, 1]$.

The limit cases $q = \pm 1$ can be considered as well, but only $q = 1$ will be allowed below.

An important class of analytic functions in the geometric function theory is the class of typically real functions [3]. This class was introduced in [7] and studied subsequently in [5,6].

Definition 1. Let $\mathscr{H}(\mathbb{D})$ denote the holomorphic functions in the unit disc $\mathbb{D}$ of the form $f(z) = z + a_2 z^2 + \cdots$.

Definition 2. A function $f(z) = z + a_2 z^2 + \cdots \in \mathscr{H}(\mathbb{D})$ is called the typically real in $\mathbb{D}$ if it satisfies the following conditions:

$$f(z) \text{ is real for real } z \in (-1, 1) \quad \text{and} \quad \operatorname{Im} f(z) \cdot \operatorname{Im} z > 0, \quad z \in \mathbb{D} \backslash (-1, 1).$$

The class of typically real functions in $\mathbb{D}$ will be denoted by $T_{\mathbb{R}}$. Robertson [6] found the integral representation for the typically real functions, namely

$$f(z) = \int_0^{\pi} \frac{z}{(1 - z\mathrm{e}^{\mathrm{i}\theta})(1 - z\mathrm{e}^{-\mathrm{i}\theta})} \, \mathrm{d}\mu(\theta) = \int_0^{\pi} \frac{z}{1 - 2z\cos\theta + z^2} \, \mathrm{d}\mu(\theta),$$

$\mu \in \mathscr{P}_{[0,\pi]}$, where $\mathscr{P}$ denotes the set of the probability measures on $[0, \pi]$. Therefore,

$$T_{\mathbb{R}} = \Big\{ f(z) = z + a_2 z^2 + \cdots \in \mathscr{H}(\mathbb{D}); \\ f(z) = \int_0^{\pi} \frac{z}{(1 - z\mathrm{e}^{\mathrm{i}\theta})(1 - z\mathrm{e}^{-\mathrm{i}\theta})} \, \mathrm{d}\mu(\theta), \ \mu \in \mathscr{P}_{[0,\pi]} \Big\}. \tag{3}$$

From the above representation, we see that the class $T_{\mathbb{R}}$ is closely connected with the generating function Ψ for the Chebyshev polynomials of the second kind, $U_n(x)$, $x = \cos\theta$, $\theta \in [0, \pi]$, [2], namely

$$\Psi(\mathrm{e}^{\mathrm{i}\theta}; z) = \frac{1}{(1 - z\mathrm{e}^{\mathrm{i}\theta})(1 - z\mathrm{e}^{-\mathrm{i}\theta})} = \sum_{n=0}^{\infty} U_n(x) z^n, \quad z \in \mathbb{D}, \tag{4}$$

where

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta, \quad \theta \in [0, \pi], \quad n = 0, 1, \ldots \ . \tag{5}$$

The form of the q-Koebe function, proposed in [4], and the kernel function in the integral representation formula (3) for the class $T_{\mathbb{R}}$ suggest the study of the following class of functions:

$$T_{\mathbb{R}}^q = \Big\{ f(z) = z + a_2 z^2 + \cdots \in \mathscr{H}(\mathbb{D}); \\ f(z) = \int_{-\pi}^{\pi} \frac{z}{(1 - z\mathrm{e}^{\mathrm{i}\theta})(1 - qz\mathrm{e}^{-\mathrm{i}\theta})} \, \mathrm{d}\mu(\theta), \ \mu \in \mathscr{P}_{[-\pi,\pi]} \Big\}. \tag{6}$$

So we see that the q-Koebe function can be used as a tool in the definition of a new class of holomorphic functions in $\mathbb{D}$.

When studying the extremal problems for $T^q_{\mathbb{R}}$, especially coefficient problems, we meet the trigonometric polynomials $U_n(q; \mathrm{e}^{\mathrm{i}\theta})$ which are defined by the generating function

$$\Psi_q(\mathrm{e}^{\mathrm{i}\theta}; z) = \frac{1}{(1 - z\mathrm{e}^{\mathrm{i}\theta})(1 - qz\mathrm{e}^{-\mathrm{i}\theta})} = \sum_{n=0}^{\infty} U_n(q; \mathrm{e}^{\mathrm{i}\theta}) z^n, \tag{7}$$

$z \in \mathbb{D}$, $\theta \in [-\pi, \pi]$, $q \in (-1, 1]$, where

$$U_n(q; \mathrm{e}^{\mathrm{i}\theta}) = \frac{\mathrm{e}^{\mathrm{i}(n+1)\theta} - q^{n+1}\mathrm{e}^{-\mathrm{i}(n+1)\theta}}{\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta}}, \quad n \geqslant 2,$$

$$U_0(q; \mathrm{e}^{\mathrm{i}\theta}) = 1, \quad U_1(q; \mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta} + q\mathrm{e}^{-\mathrm{i}\theta}. \tag{8}$$

Observe that $U_n(1; \mathrm{e}^{\mathrm{i}\theta}) = U_n(\cos\theta)$. It appears that the trigonometric polynomials $U_n(q; \mathrm{e}^{\mathrm{i}\theta})$ can be considered as the boundary values for $z = \mathrm{e}^{\mathrm{i}\theta}$ of the following symmetric Laurent polynomials:

$$U_n(q; z) = \frac{z^{n+1} - q^{n+1}/z^{n+1}}{z - q/z} = z^n + qz^{n-2} + q^2 z^{n-4} + \cdots + \frac{q^{n-1}}{z^{n-2}} + \frac{q^n}{z^n}, \quad z \neq 0,$$

$$U_n(q; z) = U_n(q; q/z). \tag{9}$$

In this paper, we study the basic properties of the trigonometric polynomials $U_n(q; \mathrm{e}^{\mathrm{i}\theta})$. The obtained results for $q = 1$ give the corresponding ones for Chebyshev polynomials of the second kind.

In particular, we find for the trigonometric polynomials $U_n(q; \mathrm{e}^{\mathrm{i}\theta})$ the three-term recurrence relation, the differential equation of the second order, the orthogonality properties, etc.

2. The main results

The function $k_1(z)$ is starlike in $\mathbb{D}$ and convex in the disc $|z| < 2 - \sqrt{3}$. The following two propositions generalize these properties for the function $k_q(z)$.

Proposition 1. *The function $k_q(z)$ is α-starlike in $\mathbb{D}$ with*

$$\alpha = \alpha(q) = \frac{1}{2}\,\frac{1 - |q|}{1 + |q|}.$$

Proof. We recall that $f \in \mathscr{H}(\mathbb{D})$ is α-starlike in $\mathbb{D}$ iff

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}, \; 0 \leqslant \alpha < 1, \; [3].$$

Taking logarithmic derivative of (2) we find

$$\frac{zk_q'(z)}{k_q(z)} = 1 + \frac{z}{1 - z} + \frac{qz}{1 - qz}.$$

Therefore for $z \in \mathbb{D}$ we have

$$\operatorname{Re}\frac{zk_q'(z)}{k_q(z)} = \frac{1}{2}\operatorname{Re}\frac{1 + z}{1 - z} + \frac{1}{2}\operatorname{Re}\frac{1 + qz}{1 - qz} \geqslant \frac{1}{2}\,\frac{1 - |q|}{1 + |q|},$$

which ends the proof. □

Proposition 2. *The function* $k_q(z)$ *is convex in the disc* $|z| < r_c(q)$, *where*

$$r_c(q) = \frac{\tau - \sqrt{\tau^2 - 4|q|}}{2|q|}$$

and $\tau = \frac{1}{2}[(1+|q|) + \sqrt{1 + 34|q| + |q|^2}]$.

Proof. A function $f \in \mathscr{H}(\mathbb{D})$ is univalent and convex in $\mathbb{D}$, if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}, \quad [3].$$

Therefore in order to find the radius $r_c(q)$ of the disc in which the q-Koebe function is convex we have to find $r \in (0, 1)$ such that

$$\min_{|z|=r<1} \operatorname{Re}\left\{1 + \frac{zk_q''(z)}{k_q'(z)}\right\}$$

is positive.

One can find

$$1 + \frac{zk_q''(z)}{k_q'(z)} = 1 + 2\frac{z}{1-z} + 2\frac{qz}{1-qz} - 2\frac{qz^2}{1-qz^2} = \frac{1+z}{1-z} + \frac{1+qz}{1-qz} - \frac{1+qz^2}{1-qz^2},$$

but

$$\frac{1-r}{1+r} \leqslant \operatorname{Re}\frac{1+z}{1-z} \leqslant \frac{1+r}{1-r}, \quad |z| = r < 1,$$

and we have

$$\operatorname{Re}\left\{1 + \frac{zk_q''(z)}{k_q'(z)}\right\} \geqslant \frac{1-r}{1+r} + \frac{1-|q|r}{1+|q|r} - \frac{1+|q|r^2}{1-|q|r^2}.$$

The last expression is positive if $0 < r < r_0$, where r_0 is the least positive root of the equation

$$|q|^2r^4 - |q|(1+|q|)r^3 - 6|q|r^2 - (1+|q|)r + 1 = 0.$$

Dividing the above equation by $|q|^2r^2$ and substituting

$$r + \frac{1}{|q|r} = t,$$

we obtain the equation

$$|q|r^2 - |q|tr + 1 = 0,$$

which implies the result. □

Proposition 3. *If $f \in T_{\mathbb{R}}^q$, then we have the following sharp bound*:

$$|a_n| \leqslant \frac{1-|q|^n}{1-|q|} = U_n(|q|;1).$$

Proof. Observe that for $f \in T_{\mathbb{R}}^q$:

$$a_n = \int_{-\pi}^{\pi} U_{n-1}(q;\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta),$$

which implies

$$\begin{aligned} |a_n| &= \int_{-\pi}^{\pi} |U_{n-1}(q;\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\mu(\theta) \leqslant \int_{-\pi}^{\pi} \left|\frac{\mathrm{e}^{\mathrm{i}n\theta} - q^n\mathrm{e}^{-\mathrm{i}n\theta}}{\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta}}\right| \mathrm{d}\mu(\theta) \\ &= \int_{-\pi}^{\pi} |\mathrm{e}^{\mathrm{i}(n-1)\theta} + q\mathrm{e}^{\mathrm{i}(n-3)\theta} + \cdots + q^{n-1}\mathrm{e}^{-\mathrm{i}(n-1)\theta}|\,\mathrm{d}\mu(\theta) \leqslant U_n(|q|;1), \end{aligned}$$

because μ is probability measure on $[-\pi,\pi]$. □

Theorem 1. *The trigonometric polynomials $U_n(q;\mathrm{e}^{\mathrm{i}\theta})$ satisfy the three-term recurrence relation*

$$U_{n+2}(q;\mathrm{e}^{\mathrm{i}\theta}) - (\mathrm{e}^{\mathrm{i}\theta} + q\mathrm{e}^{-\mathrm{i}\theta})U_{n+1}(q;\mathrm{e}^{\mathrm{i}\theta}) + qU_n(q;\mathrm{e}^{\mathrm{i}\theta}) = 0, \quad n = 0,1,\ldots \tag{10}$$

with $U_0(q;\mathrm{e}^{\mathrm{i}\theta}) = 1$, $U_1(q;\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta} + q\mathrm{e}^{-\mathrm{i}\theta}$.

Proof. Define

$$W_n = \mathrm{e}^{\mathrm{i}(n+1)\theta} - q^{n+1}\mathrm{e}^{-\mathrm{i}(n+1)\theta}. \tag{11}$$

Therefore,

$$W_{n+1} = \mathrm{e}^{\mathrm{i}(n+2)\theta} - q^{n+2}\mathrm{e}^{-\mathrm{i}(n+2)\theta} \tag{12}$$

and

$$W_{n+2} = \mathrm{e}^{\mathrm{i}(n+3)\theta} - q^{n+3}\mathrm{e}^{-\mathrm{i}(n+3)\theta}. \tag{13}$$

Multiplying (11) by $\mathrm{e}^{\mathrm{i}\theta}$ and $q\mathrm{e}^{-\mathrm{i}\theta}$ and subtracting from (12) we get

$$W_{n+1} - \mathrm{e}^{\mathrm{i}\theta}W_n = q^{n+1}\mathrm{e}^{-\mathrm{i}(n+1)\theta}(\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta}),$$

$$W_{n+1} - q\mathrm{e}^{-\mathrm{i}\theta}W_n = \mathrm{e}^{\mathrm{i}(n+1)\theta}(\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta}).$$

Finding from the above $\mathrm{e}^{\mathrm{i}(n+1)\theta}$ and $\mathrm{e}^{-\mathrm{i}(n+1)\theta}$ and putting them into (13) we obtain after short calculations

$$W_{n+2} = \mathrm{e}^{2\mathrm{i}\theta} \frac{W_{n+1} - q\mathrm{e}^{-\mathrm{i}\theta} W_n}{\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta}} - q^2 \mathrm{e}^{-2\mathrm{i}\theta} \frac{W_{n+1} - \mathrm{e}^{\mathrm{i}\theta} W_n}{\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta}},$$

which gives $W_{n+2} = (\mathrm{e}^{\mathrm{i}\theta} + q\mathrm{e}^{-\mathrm{i}\theta})W_{n+1} - qW_n$, and formula (10), which ends the proof. □

Theorem 1 can be expressed differently.

Theorem 1a. *Let p_n be the polynomial system satisfying the three-term recurrence relation*:

$$xp_n(x) = p_{n+1}(x) + qp_{n-1}(x) \tag{*}$$

with the initial conditions $p_0(x) = 1, \ p_1(x) = x$. *Then*

$$U_n(q; \mathrm{e}^{\mathrm{i}\theta}) = p_n(\mathrm{e}^{\mathrm{i}\theta} + q\mathrm{e}^{-\mathrm{i}\theta}).$$

Proof. By Favard's theorem [2] p_n form an orthogonal system on the real line. One can observe that from the three-term recurrence relation for the Chebyshev polynomials of the second kind

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x), \quad U_0(x) = 1, \ U_1(x) = 2x,$$

by substitution $x = t/2\sqrt{q}$ and rearrangement we obtain

$$tq^{n/2} U_n\left(\frac{t}{2\sqrt{q}}\right) = q^{(n+1)/2} U_{n+1}\left(\frac{t}{2\sqrt{q}}\right) + qq^{(n-1)/2} U_{n-1}\left(\frac{t}{2\sqrt{q}}\right).$$

This shows that

$$p_n(x) = q^{n/2} U_n\left(\frac{x}{2\sqrt{q}}\right).$$

Comparison of (*) and (10) implies the equality

$$U_n(q; \mathrm{e}^{\mathrm{i}\theta}) = p_n(\mathrm{e}^{\mathrm{i}\theta} + q\mathrm{e}^{-\mathrm{i}\theta}). \qquad \square$$

Theorem 2. *The function $y(\theta) = U_n(q; \mathrm{e}^{\mathrm{i}\theta})$ satisfies the following differential equation of the second order*:

$$(\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta})y'' + 2i(\mathrm{e}^{\mathrm{i}\theta} + q\mathrm{e}^{-\mathrm{i}\theta})y' + n(n+2)(\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta})y = 0. \tag{14}$$

Proof. Using the explicit formula (8) for $U_n(q; \mathrm{e}^{\mathrm{i}\theta})$ after double differentiation we obtain (14). □

Theorem 3. *The trigonometric polynomials $U_n(q; \mathrm{e}^{\mathrm{i}\theta})$ satisfy the following orthogonality relation*:

$$\int_{-\pi}^{\pi} U_n(q; \mathrm{e}^{\mathrm{i}\theta}) \overline{U}_m(q; \mathrm{e}^{\mathrm{i}\theta}) \rho(\theta) \, \mathrm{d}\theta = \begin{cases} 0 & \textit{if } m \neq n, \\ \frac{\pi}{2}(1 + q^{2(n+1)}) & \textit{if } m = n, \end{cases} \tag{15}$$

where

$$\rho(\theta) = \left| \frac{\mathrm{e}^{\mathrm{i}\theta} - q\mathrm{e}^{-\mathrm{i}\theta}}{2i} \right|^2.$$

Proof. We have

$$\begin{aligned}
&\int_{-\pi}^{\pi} U_n(q;\mathrm{e}^{\mathrm{i}\theta})\overline{U}_m(q;\mathrm{e}^{\mathrm{i}\theta})\left|\frac{\mathrm{e}^{\mathrm{i}\theta}-q\mathrm{e}^{-\mathrm{i}\theta}}{2i}\right|^2\mathrm{d}\theta\\
&=\frac{1}{4}\int_{-\pi}^{\pi}\frac{\mathrm{e}^{\mathrm{i}(n+1)\theta}-q^{n+1}\mathrm{e}^{-\mathrm{i}(n+1)\theta}}{\mathrm{e}^{\mathrm{i}\theta}-q\mathrm{e}^{-\mathrm{i}\theta}}\,\frac{\mathrm{e}^{-\mathrm{i}(m+1)\theta}-q^{m+1}\mathrm{e}^{\mathrm{i}(m+1)\theta}}{\mathrm{e}^{-\mathrm{i}\theta}-q\mathrm{e}^{\mathrm{i}\theta}}\\
&\quad Times(\mathrm{e}^{\mathrm{i}\theta}-q\mathrm{e}^{-\mathrm{i}\theta})(\mathrm{e}^{-\mathrm{i}\theta}-q\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta\\
&=\frac{1}{4}\int_{-\pi}^{\pi}(\mathrm{e}^{\mathrm{i}(n-m)\theta}-q^{m+1}\mathrm{e}^{\mathrm{i}(n+m+2)\theta}-q^{n+1}\mathrm{e}^{-\mathrm{i}(n+m+2)\theta}+q^{m+n+2}\mathrm{e}^{\mathrm{i}(m-n)\theta})\,\mathrm{d}\theta\\
&=\begin{cases}0 & \text{if } m\neq n,\\ \dfrac{\pi}{2}(1+q^{2(n+1)}) & \text{if } m=n.\end{cases}
\end{aligned}$$

Together with the trigonometric polynomials $U_n(q;\mathrm{e}^{\mathrm{i}\theta})$ one can consider the related family of trigonometric polynomials

$$T_n(q;\mathrm{e}^{\mathrm{i}\theta})=\tfrac{1}{2}(\mathrm{e}^{\mathrm{i}n\theta}+q^n\mathrm{e}^{-\mathrm{i}n\theta}), \tag{16}$$

which for $q=1$ and $\theta\in[0,\pi]$ give the Chebyshev polynomials of the first kind. □

Theorem 1′. *The trigonometric polynomials $T_n(q;\mathrm{e}^{\mathrm{i}\theta})$ satisfy the three-term recurrence relation*

$$T_{n+2}(q;\mathrm{e}^{\mathrm{i}\theta})-(\mathrm{e}^{\mathrm{i}\theta}+q\mathrm{e}^{-\mathrm{i}\theta})T_{n+1}(q;\mathrm{e}^{\mathrm{i}\theta})+qT_n(q;\mathrm{e}^{\mathrm{i}\theta})=0,\quad n=0,1,\ldots \tag{17}$$

with $T_0(q;\mathrm{e}^{\mathrm{i}\theta})=1$, $T_1(q;\mathrm{e}^{\mathrm{i}\theta})=\frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta}+q\mathrm{e}^{-\mathrm{i}\theta})$.

Remark 1. Analogously as in Theorem 1a, we may observe that

$$T_n(q;\mathrm{e}^{\mathrm{i}\theta})=q^{n/2}T_n\left(\frac{\mathrm{e}^{\mathrm{i}\theta}+q\mathrm{e}^{-\mathrm{i}\theta}}{2\sqrt{q}}\right).$$

Theorem 2′. *The function $y(\theta)=T_n(q;\mathrm{e}^{\mathrm{i}\theta})$ satisfies the following differential equation*:

$$y''+n^2y=0. \tag{18}$$

Theorem 3′. *The trigonometric polynomials $T_n(q;\mathrm{e}^{\mathrm{i}\theta})$ satisfy the following orthogonality relation*:

$$\int_{-\pi}^{\pi}T_n(q;\mathrm{e}^{\mathrm{i}\theta})\overline{T}_m(q;\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta=\begin{cases}0 & \text{if } m\neq n,\\ 2\pi & \text{if } m=n=0,\\ \dfrac{\pi}{2}(1+q^{2n}) & \text{if } m=n.\end{cases} \tag{19}$$

Remark 2. Several formulas for Chebyshev polynomials of the second and the first kind can be easily extended to the polynomials $U_n(q;\mathrm{e}^{\mathrm{i}\theta})$ and $T_n(q;\mathrm{e}^{\mathrm{i}\theta})$. Eg., the following is true ($q\in(-1,1]$),

$n = 1, 2, \ldots$:

$$U_n(q; e^{i\theta}) = \frac{-2i}{(n+1)(e^{i\theta} - q e^{-i\theta})} \, T'_{n+1}(q; e^{i\theta}),$$

$$(e^{i\theta} - q e^{-i\theta})^2 U'_n(q; e^{i\theta}) = in(e^{2i\theta} - q^2 e^{-2i\theta}) U_n(q; e^{i\theta}) - 2iq(n+1)(e^{i\theta} - q e^{-i\theta}) U_{n-1}(q; e^{i\theta}),$$

$$(e^{i\theta} - q e^{-i\theta})^2 U^2_{n-1}(q; e^{i\theta}) = 2[T_{2n}(q; e^{i\theta}) - q^n],$$

$$4[T_n(q; e^{i\theta})]^2 - (e^{i\theta} - q e^{-i\theta})^2 [U_{n-1}(q; e^{i\theta})]^2 = 4q^n.$$

Remark 3. The trigonometric polynomials $U_n(q; e^{i\theta})$ and $T_n(q; e^{i\theta})$ have the following representation:

$$U_n(q; e^{i\theta}) = \sum_{k=0}^{[n/2]} (-1)^k q^k \frac{(n-k)!}{(n-2k)!k!} (e^{i\theta} + q e^{-i\theta})^{n-2k},$$

$$T_n(q; e^{i\theta}) = \frac{n}{2} \sum_{k=0}^{[n/2]} \frac{(-1)^k q^k}{k!} \frac{(n-k-1)!}{(n-2k)!} (e^{i\theta} + q e^{-i\theta})^{n-2k}.$$

Remark 4. The obvious extension of (7) for the Gegenbauer polynomials $C_n^{\lambda}(q; e^{i\theta})$ can be done as well.

Acknowledgements

The authors would like to thank the referee for very helpful remarks which substantially improved the earlier version of this paper.

References

[1] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. 154 (1985) 137–152.
[2] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, London, Paris, 1978.
[3] P.L. Duren, Univalent Functions, Springer, Berlin, New York, 1983.
[4] G. Gasper, q-extensions of Clausen's formula and of the inequalities used by de Branges in his proof of the Bieberbach, Robertson and Milin conjecture, SIAM J. Math. Anal. 20 (1989) 1019–1034.
[5] G.M. Golusin, On typically-real functions, Math. Sb. 27 (69, 2) (1950) 201–217 (in Russian).
[6] M.S. Robertson, On the coefficient of typically-real functions, Bull. Amer. Math. Soc. 41 (1935) 565–572.
[7] W. Rogosinski, Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen, Math. Zeit. 35 (1932) 93–121.